Category of Finite Dimensional Modules over an Orthosymplectic Lie Superalgebra: Small Rank Examples

Caroline Gruson, Vera Serganova

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Caroline Gruson [1] and Vera Serganova [2]

Dedicated to M. Jimbo

Introduction

Let $\mathfrak{g}$ be a complex orthosymplectic Lie superalgebra and let G be the corresponding algebraic supergroup $SOSP(m, 2n)$. Consider the category $\mathcal{F}$ of finite dimensional G-modules such that the parity of a weight space coincides with the parity of the corresponding weight. In previous work ([7], [8]), we proved results concerning the character of simple objects in $\mathcal{F}$ and projective indecomposable modules. In particular, we showed that a Bernstein-Gel'fand-Gel'fand reciprocity law holds in $\mathcal{F}$.

The aim of this presentation is to describe the algorithms introduced in [7] and [8] in low rank examples. We start with a summary of those two papers in the case $\mathfrak{osp}(2m+1, 2n)$. We then give a complete description of the algorithms for the maximally atyical weights of $\mathfrak{osp}(5, 4)$. Using these algorithms, we are able to give multiplicities of simple modules occurring in a projective indecomposable module: up to now, such explicit computations were available only for weights of atypicality degree less or equal to 1 (here we get atypicality degree 2). In the last section, we consider the case $\mathfrak{osp}(7, 6)$, where such a complete description is rather more complicated and we draw the picture for "generic weights" (such a picture is also obtained for $\mathfrak{osp}(2n+1, 2n)$). We completely describe the "exceptional moves" for $\mathfrak{osp}(7, 6)$, this is the smallest case where these moves can start from infiniteley many weights.

We encode dominant weights by weight diagrams, following the idea of Brundan and Stroppel for the $\mathfrak{gl}(m, n)$ case ([2]). The category $\mathcal{F}$ splits into blocks, which are indexed by the core of these weight diagram. We only consider maximally atypical weights since we know that, with the help of translation functors all the other cases can be reduced to that one, see Theorem 2 in [7]. We restrict ourselves to algebras of type $\mathfrak{osp}(2m+1, 2n)$ in order to limit the notations...

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1. Context

Let us first recall a few facts about Lie superalgebras.

It is well-known that the representation theory of simple Lie superalgebras is not a straightforward adaptation of the theory in the non graded case. In 1977, Kac in [9], classified the simple Lie superalgebras, and emphazised on the fact that the finite dimensional modules are not semi-simple. When the Lie superalgebra is basic classical, the simple modules have a highest weight, which is a dominant weight for the reductive Lie algebra

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which forms the even part. He asked the question of computing the characters for simple modules and introduced the Kac modules for the case of $\mathfrak{gl}(m,n)$: there is a parabolic subalgebra $\mathfrak{p}$ with a purely odd complement space. A Kac module is obtained by inflating a simple module from the Levi part $\mathfrak{gl}(m) \times \mathfrak{gl}(n)$ of $\mathfrak{p}$ to $\mathfrak{p}$, then by inducing from $\mathfrak{p}$ to $\mathfrak{gl}(m,n)$: the induced module is still finite dimensional and there is a neat character formula for them. Moreover, Kac modules play the role of standard modules in the BGG reciprocity law in the category of finite dimensional modules, as is first mentioned in [18]. This category, for $\mathfrak{gl}(m,n)$, is now quite well understood ([15], [1], [2], [3], [4], [5]).

It is tempting to do the same with orthosymplectic superalgebras, but they have no such parabolic subalgebras, hence in this case, Kac modules no longer exist. However, one can give a geometric interpretation Borel-Weil-Bott like for Kac modules for $\mathfrak{gl}(m,n)$, as the space of sections of a line bundle over the super flag variety. Hence, one can make the corresponding construction in the $\mathfrak{osp}$ case ([14]): now the cohomology is no longer concentrated in degree 0, and as is first mentioned in [16], we introduce the *Euler characteristic* which is a virtual module in the Grothendieck group $\mathcal{K}(\mathcal{F})$ of the category defined as the alternating sum of the cohomolgy groups: we will be more precise later.

Those virtual modules stand for the standard objects for $\mathcal{F}$, meaning that they have computable composition series in terms of the simple modules ([7]), and the indecomposable projective modules can be uniquely expressed as linear combinations (with not necessarily positive integral coefficients) of Euler characteristics. Moreover, a BGG reciprocity law holds ([8]). It is to be noted that there are less standard objects than projective or simple modules, since they are labelled by weights belonging to a smaller set.

We also want to emphasize that for $\mathfrak{osp}(2m+1,2n)$, the multiplicity of a simple module in any Euler characteristic is at most 1 (but not for $\mathfrak{osp}(2m,2n)$ in general).

Now let us be a little more precise. Let $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$, we denote by $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the decomposition into even and odd parts. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ together with a basis $(\varepsilon_1,\dots,\varepsilon_m,\delta_1,\dots\delta_n)$ of $\mathfrak{h}^*$, denote by W the associated Weyl group. The roots split into the roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}$, Δ_0, and the odd roots Δ_1 are the weights of $\mathfrak{g}_1$. The Killing form on $\mathfrak{g}$ restricts to a non-degenerate bilinear form on $\mathfrak{h}$ up to a scalar, it is given by $(\varepsilon_i,\varepsilon_j) = \delta_{ij} = -(\delta_i,\delta_j)$, and $(\varepsilon_i,\delta_j) = 0$. We choose the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ (and in doing so we get a choice of positive roots), such that:

- If $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$ and $m \geq n$, the simple roots are

$$\varepsilon_1-\varepsilon_2,...,\varepsilon_{m-n+1}-\delta_1,\delta_1-\varepsilon_{m-n+2},...,\varepsilon_m-\delta_n,\delta_n,$$

$$\rho = -\frac{1}{2}\sum_{i=1}^m \varepsilon_i + \frac{1}{2}\sum_{j+1}^n \delta_j + \sum_{i=1}^{m-n}(m-n-i+1)\varepsilon_i;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$ and $m < n$, the simple roots are

$$\delta_1-\delta_2,...,\delta_{n-m}-\varepsilon_1,\varepsilon_1-\delta_{n-m+1},...,\varepsilon_m-\delta_n,\delta_n,$$

$$\rho = -\frac{1}{2}\sum_{i=1}^m \varepsilon_i + \frac{1}{2}\sum_{j+1}^n \delta_j + \sum_{j=1}^{n-m}(n-m-j)\delta_j,$$

here $\rho = \rho_0 - \rho_1$ is the graded version of half sum of positive roots, where $\rho_i = \frac{1}{2}\sum_{\alpha\in\Delta_i^+}\alpha$.

Recall (see [7] Corollary 3) that λ is the highest weight of a simple finite dimensional $\mathfrak{g}$-module (or λ is integral dominant) if and only if

$$\lambda+\rho = a_1\varepsilon_1+...+a_m\varepsilon_m+c_1\delta_1+...+c_n\delta_n,$$

where $a_i, c_j \in \frac{1}{2} + \mathbb{Z}$, and either

$$a_1 > a_2 > ... > a_m \geq \frac{1}{2}\ ,\ c_1 > c_2 > ... > c_n \geq \frac{1}{2},$$

or there exists $\ell \in \{0, \ldots, \min(m,n)\}$ such that

$$\begin{cases} a_1 > a_2 > \cdots > a_{m-\ell-1} > a_{m-\ell} = \cdots = a_m = -\frac{1}{2}, \\ c_1 > c_2 > \cdots > c_{n-\ell-1} \geq c_{n-\ell} = \cdots = c_n = \frac{1}{2}. \end{cases}$$

There is a partial ordering on the set of dominant weights, namely $\lambda \leq \mu$ iff $\mu - \lambda = \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_+$.

Moreover, recall that a weight λ is *atypical* if there exist isotropic odd root(s) α such that $(\lambda + \rho, \alpha) = 0$. The degree of atypicality is defined in Definition 2 in [7], we will explain in the next section how to compute it with the weight diagrams.

Let G be the algebraic supergroup $SOSP(2m+1, 2n)$ and Q be a parabolic subgroup containing B, the Borel subgroup of G with Lie algebra $\mathfrak{b}$. There is a structure of algebraic supervariety on the flag manifold G/Q. Let λ be a dominant weight, one can associate to λ a vector bundle $\mathcal{L}_{G/Q}(\lambda)$ over G/Q and a structure of $\mathfrak{g}$-module on the cohomology groups $H^i(G/Q, \mathcal{L}(\lambda))$. The Euler characteristic is the following virtual module:

$$\mathcal{E}_{G/Q}(\lambda) = \sum_{0 \leq i \leq dim(G/Q)} (-1)^i [H^i(G/Q, \mathcal{L}(\lambda))] \in \mathcal{K}(\mathcal{F}).$$

In most cases, the Euler characteristic mentioned above is $\mathcal{E}(\lambda) = \mathcal{E}_{G/B}(\lambda)$, but for certain weights, namely when λ has a tail (see [7] after Lemma 15 and next section), it turns out that $\mathcal{E}_{G/B}(\lambda)$ vanishes and then one finds a proper parabolic subgroup Q_λ associated to λ, such that $\mathcal{E}(\lambda) = \mathcal{E}_{G/Q_\lambda}(\lambda)$ is non-zero.

2. Summary of [7] and [8] in the $\mathfrak{osp}(2m+1, 2n)$ case

A dominant weight λ such that

$$\lambda + \rho = a_1\varepsilon_1 + ... + a_m\varepsilon_m + c_1\delta_1 + ... + c_n\delta_n$$

is encoded in the *weight diagram* denoted f_λ constructed as follows:

A weight diagram is a assignation of zero, one or several symbols $<$, $>$, or $\times$ to positions $t = \frac{2r+1}{2}$, $r \in \mathbb{Z}_{\geq 0}$, maybe endowed with a sign $(+)$ or $(-)$:

- 1) put one symbol $>$ at position t for every i such that $|a_i| = t$;
- 2) put one symbol $<$ at position t for every i such that $c_i = t$;
- 3) for every t, replace a pair of symbols $>$ and $<$, by a single $\times$, as many times as possible;
- 4) if $t = \frac{1}{2}$ and the smallest value of a_i for which $|a_i| = \frac{1}{2}$ is positive (resp. negative), put a $(+)$ (resp. $(-)$) in front of the diagram.

Remarks -

- 1) There is a one-to-one correspondence between dominant weights and weight diagrams.
- 2) Due to the dominance conditions, there is at most one symbol at a position $t \neq \frac{1}{2}$.

– 3) The *atypicality degree* of λ is by definition the maximal number of mutually orthogonal isotropic roots which are orthogonal to $\lambda+\rho$, such roots are necessarily odd, and it turns out to be the total number of $\times$ in f_λ.
– 4) The position $t=\frac{1}{2}$ can contain at most one of the symbols $>$ or $<$, and up to the maximal possible atypicality degree symbols $\times$.

Definition 1. -

– *1) The position $t=\frac{1}{2}$ is called the tail position.*
– *2) The length of the tail of a diagram (and the corresponding weight) is equal to*
the number of $\times$ at the tail position if the diagram does not have sign or the sign is $(-)$;
the number of $\times$ at the tail position minus 1 if the diagrams has sign $(+)$.
The diagram is tailless if the length of of the tail is 0.
– *3) The core of λ is the weight diagram (for a smaller rank Lie superalgebra of the same type) obtained when removing all the $\times$ of f_λ. The core determines the block of $\mathcal{F}$ containing the modules L_λ, $\mathcal{E}(\lambda)$ and P_λ. The core symbols are all the symbols $<$ and $>$.*

Theorem 1. *(* [7]*)-*

– *1) Two simple modules L_λ and L_μ belong to the same block of $\mathcal{F}$ if and only if weight diagrams of λ and μ have the same core, and therefore the same number of $\times$.*
– *2) Two blocks B_1 and B_2 of $\mathcal{F}$ are equivalent if and only if: let $L_\lambda\in B_1, L_\mu\in B_2$, f_λ and f_μ have the same number of $\times$..*

Examples 1) If $\lambda=(\frac{9}{2},\frac{7}{2},\frac{-1}{2}|\frac{7}{2},\frac{5}{2},\frac{1}{2})$, then f_λ is $(-)\times\circ<\times>\ldots$. The symbol $\circ$ stands for an empty position, all positions to the right of $>$ are empty. The atypicality degree is 2, and the length of the tail is 1.
2) If $\lambda=(\frac{9}{2},\frac{7}{2},\frac{1}{2},\frac{-1}{2}|\frac{7}{2},\frac{5}{2},\frac{1}{2},\frac{1}{2})$ then f_λ is

$$\begin{array}{cl} (-) & \times\circ<\times>\ldots \\ & \times \end{array}$$

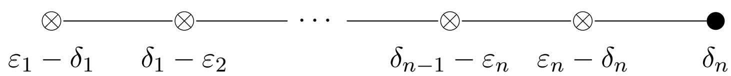
the atypicality degree is 3 and the length of the tail is 2.

Recall that the *translation functors* are functors in $\mathcal{F}$ sending a block to another one (or possibly the same one). A translation functor is a composition of tensoring with the standard representation of $\mathfrak{osp}(2m+1,2n)$ and projecting on the appropriate block. See for details [7] Section 5.

Important remark - Both papers describe algorithms giving, in the first one, the composition series of $\mathcal{E}_{G/B}(\lambda)$ or $\mathcal{E}_{G/Q_\lambda}(\lambda)$ if λ has a tail, in terms of simple modules, and in the second one an expression of a projective indecomposable as a linear combination of Euler characteristics for tailless weights, $\mathcal{E}_{G/B}(\mu)$.

2.1. **Summary of** [7] **for $\mathfrak{osp}(2n+1,2n)$.** This paper is focused on the character formula for simple modules. We restrict our attention to the maximally atypical block of $\mathfrak{osp}(2n+1,2n)$ since the translation functors lead us to understand all the other blocks, once this family of blocks is understood, see Theorem 2 and Corollary 5 in [7].

The Dynkin diagram of $\mathfrak{osp}(2n+1,2n)$ is the following:



The principle of the method is as follows: the Euler characteristics have a character which is easy to compute, so the idea is to write the composition series of the Euler characteristics in terms of simple modules. Note that the the highest weights of these simple modules are lower than the dominant weight of the Euler characteristic: thus one gets a triangular matrix with 1 on the diagonal. Inverting this matrix expresses a simple module in terms of Euler characteristics, and we deduce its character by applying the character formula for the Euler characteristics.

Let Q be a parabolic subgroup of G containing B and μ be an integral dominant weight which induces a one-dimensional representation of Q. Recall that

$$\mathcal{E}_{G/Q}(\mu) = \sum_{i=1}^{dim G_0/Q_0} (-1)^i [H^i(G/Q, \mathcal{O}(-\mu))^*].$$

If μ has a tail, then $\mathcal{E}_{G/B}(\mu) = 0$. If the length of the tail of μ is $k+1$, we define $\mathfrak{q}_\mu$ as the parabolic subalgebra containing $\mathfrak{b}$ such that the semi-simple part of its Levi subalgebra has the following Dynkin diagram:

which is the Dynkin diagram of Lie superalgebra of the same type as $\mathfrak{osp}(2n+1, 2n)$. Note that for a tailless μ, $\mathfrak{q}_\mu = \mathfrak{b}$.

As an element in the Grothendieck group of $\mathcal{F}$, the Euler characteristic $\mathcal{E}_{G/Q_\mu}(\mu)$ has a decomposition

$$\mathcal{E}_{G/Q_\mu}(\mu) = \sum a(\mu, \lambda)[L_\lambda].$$

Furthermore, $a(\mu, \mu) = 1$ and $a(\mu, \lambda) \neq 0$ implies $\lambda \leq \mu$. The main result of [7] is a combinatorial algorithm for calculating $a(\mu, \lambda)$. Below we describe this algorithm.

Since in our case λ and μ are maximally atypical, their weight diagrams don't have any core symbols.

We say f_μ is obtained from f_λ by an elementary move if one or two $\times$ of f_λ are moved to some empty positions to the right according the following rules.

- 1) Exceptional moves: can be made when λ has two $\times$ at the tail position, which are both moved simultaneously: see Definition 6, section 11 of [7] for a precise definition, see the list of exceptional moves in the following sections for $\mathfrak{osp}(5,4)$ and $\mathfrak{osp}(7,6)$.
- 2) Legal moves (resp. legal tail moves): take a $\times$ of f_λ at position s, $s \neq 1/2$ (resp. $s = 1/2$), move it to the right to an empty postion $t > s$ of f_λ and obtain a new diagram f_μ. The $\times$ starts with 1 life (resp. 2 times the number of $\times$ at the tail position of f_μ), it looses 1 life going over an empty position, it gains one life over a $\times$ and should never have a negative number of lives. The number of lives that this moving $\times$ has at position t is called the degree (or the weight) of the corresponding legal move.

We say that f_μ is obtained from f_λ be a decreasing sequence of elementary moves $\lambda = \mu^0 \to \mu^1 \to \cdots \to \mu^k = \mu$ if f_{μ^i} is obtained from $f_{\mu^{i-1}}$ by moving a $\times$ to position t_i by a legal (or legal tail) move or two $\times$ to positions $s_i < t_i$ by an exceptional move and we have $t_1 > t_2 > \cdots > t_k$. The degree $l(\gamma)$ of a decreasing sequence γ of elementary moves is the sum of the degrees of the elementary moves included in the sequence.

Theorem 3 in [7] states that

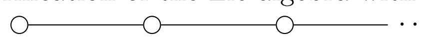
$$a(\mu, \lambda) = \sum_{\gamma \in S(\lambda,\mu)} (-1)^{l(\gamma)},$$

where the summation is taken over the set $S(\lambda, \mu)$ of all decreasing sequences of elementary moves from λ to μ.

Remark - It is proven in [7] that $a(\mu, \lambda) = \pm 1$ or 0 for all dominant integral λ, μ.

2.2. **Summary of** [8] **for** $\mathfrak{osp}(2m + 1, 2n)$**.** This second paper contains several results. First of all, it explains in a more general context that a Bernstein-Gel'fand-Gel'fand reciprocity law holds in the category $\mathcal{F}$, in other words the multiplicity of a simple module L_λ in the Euler characteristic $\mathcal{E}_{G/B}(\mu)$ is the same as the multiplicity of $\mathcal{E}_{G/B}(\mu)$ in the projective indecomposable module P_λ, this equality holding in the Grothendieck ring of $\mathcal{F}$: it is to be noted that, in this paper, the only flag variety involved is G/B.

It also contains a categorification of the Lie algebra with Dynkin diagram



in orthosymplectic terms which allows us to interpret most of the translation functors as linear operators satisfying Serre relations.

The result we are interested in for this survey is the fact that one can express any projective indecomposable module as a linear combination with integral coefficients of Euler characteristics of tailless weights. Caution, these coefficients might be negative. We explain an algorithm on the weight diagrams which gives this combination.

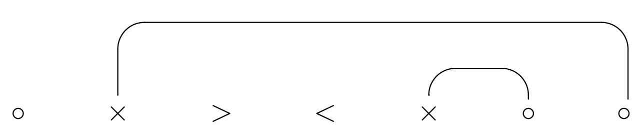
Start with a tailless dominant weight λ, and consider its weight diagram. Construct the *cap diagram* as follows:

consider the rightmost $\times$ of f_λ and join it to the next free position on the right. This position is no longer free. Repeat for the next $\times$ on the left, and so on until there is no $\times$ left. Leave all the symbols corresponding to the core where they are.

Example For the following weight diagram,

$$\circ \quad \times \quad > \quad < \quad \times \quad \circ \quad \circ$$

the caps are the following:



Denote by $\mathcal{P}(\lambda)$ the set

$\mathcal{P}(\lambda) := \{\mu$ dominant, f_μ is obtained from f_λ by moving 0 or any number of $\times$ along the caps$\}$.

Now assume that λ has a tail: we construct a tailless weight $\bar{\lambda}$ the following way:

Ignore the sign before the diagram if it exists. In the beginning, forget about the tail position of f_λ and draw the corresponding cap diagram. Then circle the $\times$, getting $\otimes$, at the tail position, and move them according to the following rules:

if λ has no core symbol at $\frac{1}{2}$ move all the $\otimes$ but one at the tail position to the free positions number 2, 4, 6, etc...

if λ has a core symbol at $\frac{1}{2}$, then move all the $\otimes$ at the tail position to the free positions number 1, 3, 5, etc...

Now draw the cap diagram of this new weight $\bar{\lambda}$.

We are now ready to state the result:

Theorem 2. -

- *1) If λ is tailless, then one has*

$$P_\lambda = \sum_{\mu \in \mathcal{P}(\lambda)} \mathcal{E}_{G/B}(\mu).$$

- *2) If f_λ has a core symbol or a $(-)$ sign,*

$$P_\lambda = \sum_{\mu \in \mathcal{P}(\bar{\lambda})} (-1)^{c(\lambda,\mu)} \mathcal{E}_{G/B}(\mu),$$

  where $c(\lambda, \mu)$ is the number of $\otimes$ in λ plus the number of $\otimes$ in $f_{\bar{\lambda}}$ moved along a cap in order to get f_μ from $f_{\bar{\lambda}}$.

- *3) If the sign before f_λ is $(+)$, use the preceding formula and change the sign of all the $\mathcal{E}_{G/B}(\mu)$ such that f_μ has a symbol at the tail position.*

The proof of this result involves a massive use of translation functors.

3. Computing characters for a simple maximally atypical module over $\mathfrak{osp}(5, 4)$

From now on, for any dominant λ we will abuse notation and set $\mathcal{E}(\lambda)$ for $\mathcal{E}_{G/Q_\lambda}(\lambda)$ if λ has a tail and $\mathcal{E}_{G/B}(\lambda)$ if λ is tailless.

In this case, a dominant weight has the form:

$$\lambda + \rho = (a_1, a_2 | c_1, c_2)$$

with $a_1 > a_2 \geq -\frac{1}{2}$ or $a_1 = a_2 = -\frac{1}{2}$ and $c_1 > c_2 \geq \frac{1}{2}$ or $c_1 = c_2 = \frac{1}{2}$. It is maximally atypical iff $|a_1| = c_1$ and $|a_2| = c_2$. The weight diagram of a maximally atypical weight contains two $\times$, one at $|a_1|$ and the other at $|a_2|$, together with a sign. If there are two $\times$ at the tail position or one $\times$ and a $(-)$ sign, then the weight has a tail and the parabolic subgroup Q_λ of the previous section is obtained by adding the opposite of the roots $\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2$ unless the weight is trivial in which case $Q_\lambda = G$. Another difficulty occurs when one gets close to the *wall* $a_1 = a_2 + 1$.

In [7] section 11, we described a series of moves which can be made with the $\times$ of the weight diagram: if there is a (authorised) move from the weight diagram f_λ to the weight diagram f_μ of weight (or degree) i, it means that the simple module L_λ is in the cohomology group of degree i corresponding to the Euler characteristic $\mathcal{E}(\mu)$, so that it occurs with the sign $(-1)^i$ in the composition series. Nevertheless, it doesn't mean that L_λ appears in $\mathcal{E}(\mu)$ because one also has to consider paths, which are sequences of moves, and it can lead to cancellations.

There are several kinds of moves: *regular* ones, which take a $\times$ at a non-tail position and move it to the right according to specific rules, *tail moves*, which deal with one $\times$

at the tail position, and *exceptional moves* which move simultaneously two $\times$ at the tail position (see Proposition 6 in [7]), in this case there is no exceptional moves.

One can check by hand all the possibilites which occur.

In the following figure 1, we have represented a maximally atypical weight $\lambda + \rho = (a_1, a_2||a_1|, |a_2|)$ by the point (a_1, a_2) in the plane and we join two points if there exists a legal move taking the weight diagram of the first weight to the weight diagram of the second one. We have equipped all the arrows with their weights.

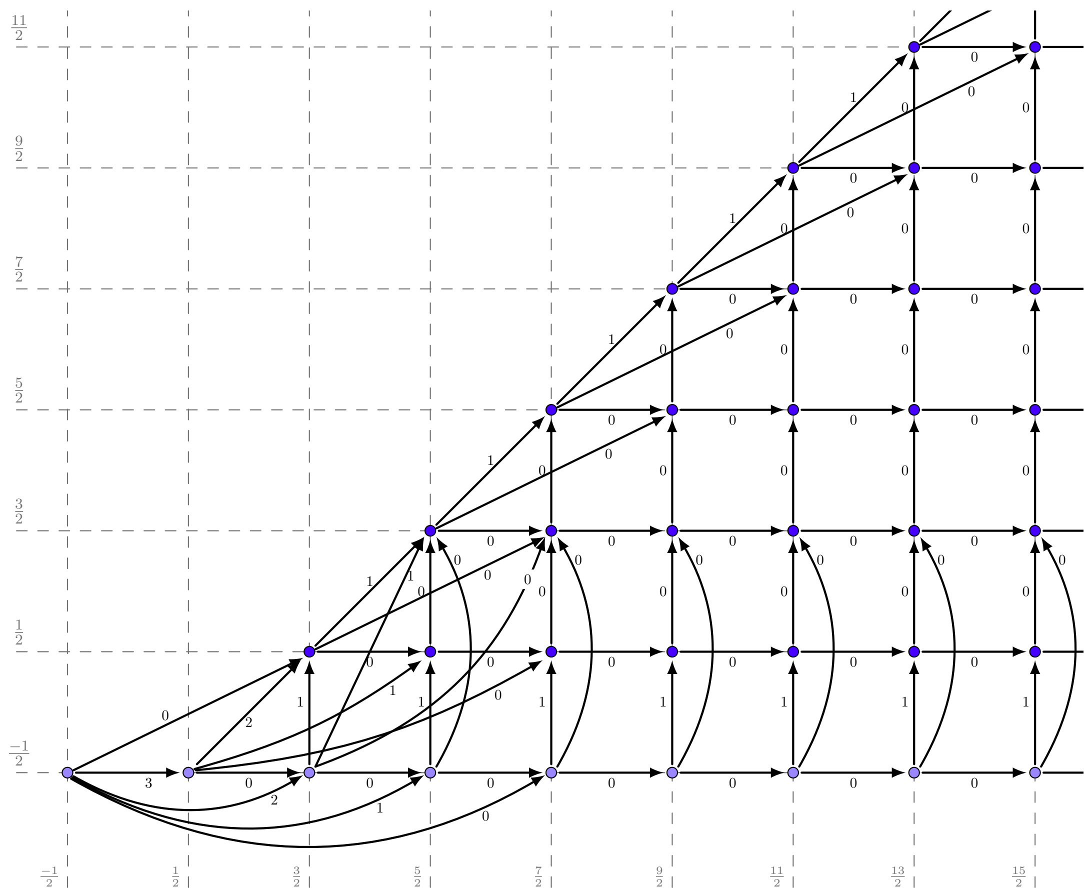


FIGURE 1. $\mathfrak{osp}(5,4)$

Now, we want to compute the multiplicity of the simple module L_λ in the Euler characteristic $\mathcal{E}_{G/Q_\mu}(\mu)$. We have to consider:

- 1) Arrow going from λ to μ with weight i (there is at most one), we will say we have a *path of length one* P and weight $wt(P) := i$.
- 2) *regular paths of length two* from λ to μ: a regular path P_i is a sequence of two arrows, one, f_1 of weight i_1, from λ to a certain λ_1 and one, f_2 with weight i_2, from λ_1 to μ, such that the first one f_1 is going East or North-East in the picture (meaning that this arrow can increase the horizontal coordinate and possibly the vertical one) and the second one f_2 goes straight North (so the horizontal coordinate cannot be increased). The weight of the corresponding path is $wt(P_i) := i_1 + i_2$.

Proposition 1. - *Let λ and μ be two dominant weights such that $\lambda \leq \mu$. Then the multiplicities are as follows:*

- *1)* $[\mathcal{E}(\mu) : L_\lambda] = 1$ *if* $\lambda = \mu$,
- *2) If* $\lambda < \mu$, *look at all the paths of length one and two from* λ *to* μ, *denote this set* $P(\lambda, \mu)$,
  $[\mathcal{E}(\mu) : L_\lambda] = \sum_{P \in P(\lambda,\mu)} (-1)^{wt(P)}$.

One can check on the picture that the module of the multiplicity of L_λ in the Euler characteristic $\mathcal{E}_{G/_\mu}(\mu)$ is at most one. This is a general phenomenon for algebras $\mathfrak{osp}(2n+1, 2m)$.

Remarks -

- 1) If μ is far from the walls, meaning that $a_1 \geq a_2 + 3 \geq 5/2$, then the partial picture explaining which L_λs appear in the Euler characteristic is just a square of size 1×1.

  If one looks at the same picture for maximally atypical weights of $\mathfrak{osp}(2n+1, 2n)$, if the weight diagram of μ has two empty positions between each couple of $\times$ and μ is far enough from the tail, the weights λ occurring in the Euler characteristic are the vertices of the hypercube with "greater vertex" μ. See figure 2 for the case $\mathfrak{osp}(7, 6)$.
- 2) As long as μ is far enough from the origin, $a_1 > 9/2$, the pattern along the walls is always the same.

We put all this information in a big (infinite) triangular unipotent matrix M labelled by all dominant maximally atypical weights, the line labelled by the weight λ encoding in which Euler characteristics L_λ occurs, and with which multiplicity. This matrix gives us the composition series of all the Euler characteristics, and since we know the character of the Euler characteristics, if we invert M and hence obtain a simple module as a linear combination of Euler characteristics, we are able to compute the character of the simple module. Well, able might be abusing language, since no one wants to explicitely compute all this...

Let us show the matrix (Table 1) for small weights, with the conventions of the figure 1 for the weights.

Table 1. Partial decomposition matrix

	$(\frac{-1}{2},\frac{-1}{2})$	$(\frac{1}{2},\frac{-1}{2})$	$(\frac{3}{2},\frac{-1}{2})$	$(\frac{5}{2},\frac{-1}{2})$	$(\frac{7}{2},\frac{-1}{2})$	$(\frac{9}{2},\frac{-1}{2})$	$(\frac{3}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{1}{2})$	$(\frac{7}{2},\frac{1}{2})$	$(\frac{9}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{3}{2})$	$(\frac{9}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{5}{2})$	$(\frac{9}{2},\frac{5}{2})$	$(\frac{9}{2},\frac{7}{2})$
$(\frac{-1}{2},\frac{-1}{2})$	1	−1	1	−1	1	0	0	1	−1	0	−1	1	0	0	0	0
$(\frac{1}{2},\frac{-1}{2})$		1	1	0	0	0	0	−1	1	0	−1	1	0	0	0	0
$(\frac{3}{2},\frac{-1}{2})$			1	1	0	0	−1	−1	0	0	0	1	0	1	0	0
$(\frac{5}{2},\frac{-1}{2})$				1	1	0	0	−1	−1	0	1	1	0	0	0	0
$(\frac{7}{2},\frac{-1}{2})$					1	1	0	0	−1	−1	0	1	1	0	0	0
$(\frac{9}{2},\frac{-1}{2})$						1	0	0	0	−1	0	0	1	0	0	0
$(\frac{3}{2},\frac{1}{2})$							1	1	0	0	0	1	0	1	0	0
$(\frac{5}{2},\frac{1}{2})$								1	1	0	1	1	0	0	0	0
$(\frac{7}{2},\frac{1}{2})$									1	1	0	1	1	0	0	0
$(\frac{9}{2},\frac{1}{2})$										1	0	0	1	0	0	0
$(\frac{5}{2},\frac{3}{2})$											1	1	0	0	1	1
$(\frac{7}{2},\frac{3}{2})$												1	1	1	1	0
$(\frac{9}{2},\frac{3}{2})$													1	0	1	0
$(\frac{7}{2},\frac{5}{2})$														1	1	0
$(\frac{9}{2},\frac{5}{2})$															1	1
$(\frac{9}{2},\frac{7}{2})$																1

For instance, let us explain how we get the column corresponding to $(\frac{3}{2}, \frac{1}{2})$: look at the picture, and the arrows coming to this weight: one gets $(\frac{3}{2}, -\frac{1}{2})$ with weight -1, then $(\frac{1}{2}, -\frac{1}{2})$ with weight 2, but it cancels with the path $(\frac{1}{2}, -\frac{1}{2}) \to (\frac{3}{2}, -\frac{1}{2}) \to (\frac{3}{2}, \frac{1}{2})$ which is of weight 1, and then the path $(-\frac{1}{2}, -\frac{1}{2}) \to (\frac{1}{2}, -\frac{1}{2}) \to (\frac{3}{2}, \frac{1}{2})$ which has weight 5 cancels with the arrow coming from the exceptional move $(-\frac{1}{2}, -\frac{1}{2}) \to (\frac{3}{2}, \frac{1}{2})$. Finally, $(\frac{3}{2}, \frac{1}{2})$ itself appears with multiplicity 1, hence the column.

4. Projective indecomposable modules for $\mathfrak{osp}(5,4)$, maximally atypical case

In [8], we showed that one can express any projective indecomposable module in the Grothendieck group $\mathcal{K}(\mathcal{F})$ as a linear combination with integral coefficients (possibly negative) of Euler characteristic for tailless weights, hence the underlying algebraic supermanifold is the flag variety G/B. We also showed that there is a (weak version of) Bernstein-Gel'fand-Gel'fand reciprocity law (see [8], Theorem 1):

Proposition 2. - *Let λ and μ be two dominant weights such that μ is tailless, one has:*

$$[\mathcal{E}(\mu) : L_\lambda] = [P_\lambda : \mathcal{E}(\mu)].$$

Remark - Note that Euler characteristics for tailless weights do not form a basis in the Grothendieck group. Since our category has infinite cohomological dimension, classes of projective modules generate a proper subgroup in the Grothendieck group (see [8]). However, Euler characteristics are linearly independent, hence the presentation of the class of a projective module as a combination of Euler characteristics is unique.

Hence, actually we have already computed all the coefficients of this linear combination while computing the characters of simple modules, or, more appropriately, the multiplicity of the simple modules occurring in a given Euler characteristic for tailless weights. Note that the (partial) matrix of the previous section contains the information for Euler characteristics for weights with a tail (the lines corresponding to weights with first coordinate equal to zero), and these ones are not relevant in the computation we do now.

Thanks to the algorithm described in [8] that allows to compute the coefficients of the linear combination of Euler characteristics involved in a given projective module, we obtain the decomposition numbers of the previous section by an independant method.

Let us take the opportunity of this paper to describe the decomposition of projective indecomposable modules of maximally atypicality degree in terms of simple modules.

Let λ be a dominant weight, we write $\lambda + \rho = (a_1, a_2 || |a_1|, |a_2|)$. For simplicity we encode λ by (a_1, a_2), as in the previous section. Assume that $a_1 - a_2 \geq 4$ and $a_2 \geq \frac{5}{2}$, we say that λ is *generic*, then the Euler characteristics involved are these of (a_1, a_2), $(a_1 + 1, a_2)$, $(a_1, a_2 + 1)$ and $(a_1 + 1, a_2 + 1)$ so that the simple modules involved are (see Table 2).

Let us study now the generic weights which are near the oblique wall.

Case $a_1 = a_2 + 3$, $a_2 \geq \frac{5}{2}$: The Euler characteristics involved are the same as in the generic case, but $\mathcal{E}(a_1, a_2 + 1)$ has $L_{(a_1-2,a_2)}$ as an additional composition factor. Hence Table 3.

Case $a_1 = a_2 + 2$: The Euler characteristics involved are the same as in the generic case, but $\mathcal{E}(a_1, a_2)$ has an additional composition factor which is $L_{(a_1-2,a_2-1)}$, $\mathcal{E}(a_1 + 1, a_2 + 1)$ has $L_{(a_1-1,a_2)}$ as an additional composition factor and $\mathcal{E}(a_1, a_2 + 1)$ is smaller than expected since it lacks $L_{(a_1-1,a_2+1)}$. Hence Table 4.

Case $a_1 = a_2 + 1$: The Euler characteristics involved are these corresponding to (a_1, a_2), $(a_1 + 1, a_2)$, $(a_1 + 2, a_2 + 1)$, $(a_1 + 2, a_2 + 2)$. We get the following Table 5.

TABLE 2. Highest weights of simple modules occurring in P_λ, λ represented by (a_1, a_2) generic

coordinates of simple factor	multiplicity
(a_1+1, a_2+1)	1
(a_1+1, a_2)	2
(a_1+1, a_2-1)	1
(a_1, a_2+1)	2
(a_1, a_2)	4
(a_1, a_2-1)	2
(a_1-1, a_2+1)	1
(a_1-1, a_2)	2
(a_1-1, a_2-1)	1

TABLE 3. Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$, $a_1 - a_2 = 3$, $a_2 \geq \frac{5}{2}$

coordinates of simple factor	multiplicity
(a_1+1, a_2+1)	1
(a_1+1, a_2)	2
(a_1+1, a_2-1)	1
(a_1, a_2+1)	2
(a_1, a_2)	4
(a_1, a_2-1)	2
(a_1-2, a_2)	1
(a_1-1, a_2+1)	1
(a_1-1, a_2)	2
(a_1-1, a_2-1)	1

TABLE 4. Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$, $a_1 - a_2 = 2$, $a_1 \geq \frac{5}{2}$

coordinates of simple factor	multiplicity
(a_1+1, a_2+1)	1
(a_1, a_2+1)	2
(a_1+1, a_2)	2
(a_1, a_2)	4
(a_1-1, a_2)	2
(a_1+1, a_2-1)	1
(a_1, a_2-1)	2
(a_1-1, a_2-1)	1
(a_1-2, a_2-1)	1

Next we study **generic weights near the tail** $a_1 \geq \frac{11}{2}$.

Case $a_2 = \frac{3}{2}$.

The Euler characteristics involved are the usual ones and we have several additional composition factors in them, see Table 6.

Case $a_2 = \frac{1}{2}$.

TABLE 5. Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$, $a_1 - a_2 = 1$, $a_2 \geq \frac{5}{2}$

coordinates of simple factor	multiplicity
(a_1+2, a_2+2)	1
(a_1+2, a_2+1)	2
(a_1+1, a_2+1)	1
(a_1+2, a_2)	1
(a_1+1, a_2)	2
(a_1, a_2)	4
(a_1+1, a_2-1)	1
(a_1, a_2-1)	2
(a_1-1, a_2-1)	1

TABLE 6. Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, 3/2)$, $a_1 \geq 11/2$

coordinates of simple factor	multiplicity
$(a_1+1, 5/2)$	1
$(a_1, 5/2)$	2
$(a_1-1, 5/2)$	1
$(a_1+1, 3/2)$	2
$(a_1, 3/2)$	4
$(a_1-1, 3/2)$	2
$(a_1+1, 1/2)$	1
$(a_1, 1/2)$	2
$(a_1-1, 1/2)$	1
$(a_1+1, -1/2)$	1
$(a_1, -1/2)$	2
$(a_1-1, -1/2)$	1

The Euler characteristics involved are the usual ones and we have several additional composition factors in them. See Table 7.

TABLE 7. Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, \frac{1}{2})$, $a_1 \geq \frac{9}{2}$

coordinates of simple factor	multiplicity
$(a_1+1, 3/2)$	1
$(a_1, 3/2)$	2
$(a_1-1, 3/2)$	1
$(a_1+1, 1/2)$	2
$(a_1, 1/2)$	4
$(a_1-1, 1/2)$	2

Case $a_1 \geq \frac{9}{2}$, $a_2 = -\frac{1}{2}$.
See Table 8.
The remaining weights are represented by the couples $(-\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(\frac{3}{2}, -\frac{1}{2})$, $(\frac{5}{2}, -\frac{1}{2})$, $(\frac{7}{2}, -\frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{5}{2}, \frac{1}{2})$, $(\frac{7}{2}, \frac{1}{2})$, $(\frac{5}{2}, \frac{3}{2})$, $(\frac{7}{2}, \frac{3}{2})$ and $(\frac{9}{2}, \frac{3}{2})$.

TABLE 8. Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, -\frac{1}{2})$, $a_1 \geq \frac{9}{2}$

coordinates of simple factor	multiplicity
$(a_1 + 1, 3/2)$	1
$(a_1, 3/2)$	2
$(a_1 - 1, 3/2)$	1
$(a_1 + 1, -1/2)$	2
$(a_1, -1/2)$	4
$(a_1 - 1, -1/2)$	2

We intend to use the partial matrix A we wrote in the previous section, suppressing the lines corresponding to Euler characteristics for weights with tail, and compute ${}^tA.A$. Caution, the relevant information in this matrix concerns only the weights which are labelled by (a_1, a_2) with $a_1 < 9/2$ and $a_2 < 5/2$, since we need additional information to get the other weights. We first do by hand the case $(a_1, a_2) = (9/2, 3/2)$, see Table 9.

TABLE 9. Highest weights of simple modules occurring in P_λ, $\lambda(9/2, 3/2)$

coordinates of simple factor	multiplicity
$(11/2, 5/2)$	1
$(9/2, 5/2)$	2
$(7/2, 5/2)$	1
$(11/2, 3/2)$	2
$(9/2, 3/2)$	4
$(7/2, 3/2)$	2
$(5/2, 3/2)$	1
$(11/2, 1/2)$	1
$(9/2, 1/2)$	2
$(7/2, 1/2)$	1
$(11/2, -1/2)$	1
$(9/2, -1/2)$	2
$(7/2, -1/2)$	1

The following Table 10 is the result of the multiplication of matrices mentioned above, it should be read this way: the line labelled by (b_1, b_2) is the decomposition of the corresponding indecomposable projective module in terms of the simple modules labelled by the colums.

5. GENERIC PICTURE FOR $\mathfrak{osp}(7, 6)$, EXCEPTIONAL MOVES FOR $\mathfrak{osp}(7, 6)$, (AND REMARKS ON HIGHER RANK CASES)

As is explained in [7], in order to get rid of the signs of the weight diagrams, it is better to look at the dominant weights of $\mathfrak{osp}(7, 8)$ belonging to the same block as the trivial module. This means adding a $<$ at the tail position, move all $\times$ not at the tail one position to the right and for the $\times$ at the tail, if the sign is $(-)$ don't change anything, whether if the sign is $(+)$ move exactly one $\times$ from the tail one position to the right.

The weight diagram of a dominant maximally atypical weight has exactly three $\times$ plus a $<$ at the tail.

Table 10. Partial Cartan matrix...

	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{3}{2},-\frac{1}{2})$	$(\frac{5}{2},-\frac{1}{2})$	$(\frac{7}{2},-\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{1}{2})$	$(\frac{7}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{3}{2})$
$(-1/2,-1/2)$	4				2	2				1
$(1/2,-1/2)$		4	2					2		1
$(3/2,-1/2)$		2	4	2	1			1	1	2
$(5/2,-1/2)$			2	4	2				2	1
$(7/2,-1/2)$	2		1	2	4	1			1	2
$(9/2,-1/2)$					2					1
$(3/2,1/2)$	2				1	4	2	1	1	2
$(5/2,1/2)$						2	4	2	2	1
$(7/2,1/2)$		2	1			1	2	4	1	2
$(9/2,1/2)$								2		1
$(5/2,3/2)$			1	2	1	1	2	1	4	2
$(7/2,3/2)$	1	1	2	1	2	2	1	2	2	4
$(9/2,3/2)$					1			1	1	2
$(7/2,5/2)$			1			1			1	2
$(9/2,5/2)$									2	1
$(9/2,7/2)$									1	

5.1. **Generic maximally atypical weights.** One can draw a picture similar to the figure 1, but it is 3-dimensional and quite intricate near the origin... Nevertheless, for a "generic" maximally atypical weight (meaning there are at least 2 empty positions between two $\times$ and it is far from the tail), the picture is easy to make, see figure 2. In this picture, the legal way is to go East then North then North-East.

Remark - For maximally atypical weights of $\mathfrak{osp}(2n+1,2n)$ wich are generic, i.e. such that the first $\times$ in the weight diagram is far from the tail position and there are at least two empty positions between two $\times$, the picture looks the same and the legal way is to move along the basis vectors corresponding first to the rightmost $\times$, then the following rightmost $\times$ and so on.

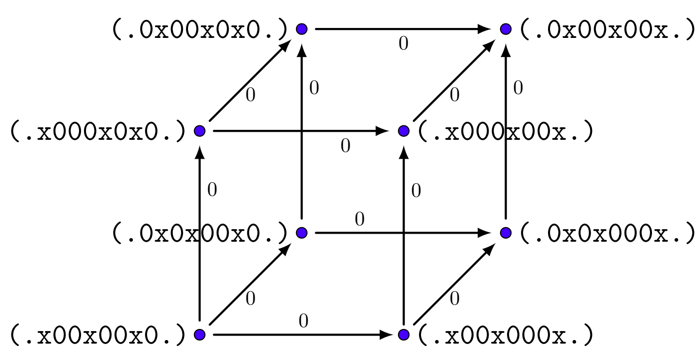


Figure 2. $\mathfrak{osp}(7,6)$, generic case

5.2. **Exceptional moves.** In $\mathfrak{osp}(7,6)$, there are infinitely many weights leading to exceptional moves, because there are more than two $\times$, see the case 5) where the rightmost

$\times$ can be at any place further right. Here is a list of these moves, we indicate the parity of the weight of the corresponding move if it is not 0.

1)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \\ \times \end{array} \longrightarrow f_\lambda = \begin{cases} < \\ \times \quad \times \quad \times \\ \text{or} \\ < \\ \times \quad \circ \quad \circ \quad \times \quad \times \end{cases}$$

2)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \quad \times \end{array} \longrightarrow f_\lambda = \; < \quad \times \quad \circ \quad \times \quad \times$$

3)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \quad \circ \quad \times \end{array} \longrightarrow f_\lambda = \begin{cases} < \quad \times \quad \times \quad \times \quad (1) \\ \text{or} \\ < \quad \times \quad \times \quad \circ \quad \times \\ or \\ < \quad \circ \quad \times \quad \times \quad \times \end{cases}$$

4)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \quad \circ \quad \circ \quad \times \end{array} \longrightarrow f_\lambda = \begin{cases} < \quad \times \quad \times \quad \times \\ \text{or} \\ < \quad \times \quad \circ \quad \times \quad \times \end{cases}$$

5)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \quad \circ \quad \circ \quad \circ \quad \times \end{array} \longrightarrow f_\lambda = \; < \quad \times \quad \times \quad \circ \quad \times \ \text{etc.}$$

Remark -This last move can be reproduced for any diagram f_μ with the same pattern at the tail and the last $\times$ at any position further on the right, with the obvious change on the diagram f_λ.

If one looks closely at the definition of admissible paths, such a move can be combined with any move concerning the $\times$ not involved in the exceptional move, so that these exceptional things are really annoying... and one has to be extremely careful in the computations. Is there still anyone wondering why we didn't draw the complete figure?

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